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C^k-RESOLUTION OF SEMIALGEBRAIC MAPPINGS. ADDENDUM TO *VOLUME GROWTH AND ENTROPY*

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ABSTRACT

We prove that a bounded semialgebraic function can be (piecewise) reparametrized in such a way that all the derivatives up to a fixed order k, with respect to new coordinates, are small, and the number of pieces is effectively bounded.

In the preceding paper [3], *Volume growth and entropy*, the proof of the main theorems 1.4 and 2.1 was given only for the dimension l of the simplex, equal to one or two.

Here we complete the proof for any l. In fact we prove a stronger (and sharp) inequality: for $f: N \to N$, $f \in C^k$,

$$v_{l,k}(f) \leqq h(f) + \frac{l}{k} R(f),$$

where $2l/k$ of Theorem 1.4 is replaced by l/k (and respectively in Theorem 2.1).

The main result here concerns the possibility to reparametrize a given bounded semialgebraic function in such a way that all the derivatives up to a fixed order k, with respect to new coordinates, are small.

Since this result seems to be of independent interest, we try to introduce an appropriate terminology, and to prove a result in a reasonably general form.

DEFINITION 1. A semialgebraic set $A \subseteq \mathbf{R}^m$ is the set representable as a result of a finite number of set-theoretic operations over sets of the form $\{g_i = 0\}$, $\{g_j > 0\}$, where g_i, g_j are the polynomials.

A set-theoretic formula of this representation together with the dimension m and the degrees of polynomials g_i, g_j is called the diagram of this representation.

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In what follows, the diagram $d(A)$ always means "the diagram of some representation of A".

It is well known that all the usual set-theoretic and topological operations with semialgebraic sets lead to semialgebraic sets with the diagrams depending only on the diagrams of the initial sets (see e.g. [1]).

DEFINITION 2. A semialgebraic mapping $u: \mathbf{R}^l \to \mathbf{R}^m$ is a closed semialgebraic subset $\Gamma_u \subseteq \mathbf{R}^l \times \mathbf{R}^m$. The diagram $d(u)$ is defined as $d(\Gamma_u)$.

The domain of definition $D_u \subseteq \mathbf{R}^l$ is $\pi(\Gamma_u)$, where $\pi: \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^l$.

The norm $\|u\|$ is defined as $\sup_{(x,y)\in D_u} \|y\|$.

Below we use compositions of semialgebraic mapping, which in our general setting can be defined as follows:

DEFINITION 3. Let $u: \mathbf{R}^l \to \mathbf{R}^m$, $v: \mathbf{R}^m \to \mathbf{R}^p$ be semialgebraic mappings.

In $\mathbf{R}^l \times \mathbf{R}^m \times \mathbf{R}^p$ consider $\Gamma'_u = \Gamma_u \times \mathbf{R}^p$, and $\Gamma'_v = \mathbf{R}^l \times \Gamma_v$. Then the composition $v \circ u: \mathbf{R}^l \to \mathbf{R}^p$ is defined by

$$\Gamma_{v \circ u} = \pi(\Gamma'_u \cap \Gamma'_y) \subseteq \mathbf{R}^l \times \mathbf{R}^p, \qquad \text{where } \pi: \mathbf{R}^l \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}^l \times \mathbf{R}^p.$$

By definition,

$$\Gamma_{v \circ u} = \{(x,z) \in \mathbf{R}^l \times \mathbf{R}^p \mid \exists y \in \mathbf{R}^m \text{ such that } (x,y) \in \Gamma_u, (y,z) \in \Gamma_v\},$$

so for usual mappings u, v, $v \circ u$ is a usual composition.

Clearly, $d(v \circ u)$ depends only on $d(u)$ and $d(v)$.

Below we always assume that semialgebraic mappings considered satisfy the following additional restriction: for $u: \mathbf{R}^l \to \mathbf{R}^m$, $\dim \Gamma_u = l$.

Let $\mathbf{Q}^l \subseteq \mathbf{R}^l$ be the unit cube $[0,1]^l$. For $\rho > 0$ let

$$\mathbf{Q}^l_\rho = [0,\rho] \times [0,1]^{l-1} \subseteq \mathbf{R}^l,$$

or, more generally,

$$\mathbf{Q}^l_\rho = [0,1]^q \times [0,\rho] \times [0,1]^{l-q-1}, \qquad q = 0,1,\dots,l-1.$$

(In what follows the order of variables is important.)

DEFINITION 4. C^k-reparametrization (k-rep.) is a semialgebraic mapping $\psi: \mathbf{R}^m \to \mathbf{R}^m$, which is an analytic diffeomorphism of some neighbourhood of $\mathbf{Q}^l$, and

(1) $\psi(\mathbf{Q}^l) \subseteq \mathbf{Q}^l$,

(2) $\|d^s\psi\|_{\mathbf{Q}^l} \leqq 1$, $s = 1,\dots,k$.

For $\rho > 0$, ρ-small C^k reparametrization ((ρ, k)-rep.) is a semialgebraic mapping $\psi: \mathbf{R}^l \to \mathbf{R}^l$, which is an analytic diffeomorphism of some neighbourhood of $\mathbf{Q}^l_\rho$, and

(1) $\psi(\mathbf{Q}^l_\rho) \subseteq \mathbf{Q}^l$,

(2) $\| d^s \psi \|_{\mathbf{Q}^l_\rho} \leqq 1$, $s = 1, \ldots, k$.

We say that the mapping $\psi: \mathbf{R}^l \to \mathbf{R}^l$ satisfies condition (*) if for each $i = 1, \ldots, l$, $\psi(\mathbf{R}^i \times z) \subseteq \mathbf{R}^i \times z'$, $z, z' \in \mathbf{R}^{l-i}$. Here

$$\mathbf{R}^l = \overbrace{\mathbf{R} \times \cdots \times \mathbf{R}}^{l}$$

is identified with

$$\mathbf{R}^i \times \mathbf{R}^l = \overbrace{\mathbf{R} \times \cdots \times \mathbf{R}}^{i} \times \overbrace{\mathbf{R} \times \cdots \times \mathbf{R}}^{l-i}.$$

Clearly, (*) is preserved by compositions. Reparametrizations as above, satisfying (*), are denoted by $(*, k)$-rep. and $(*, \rho, k)$-rep., respectively.

Let $u: \mathbf{Q}^l \to \mathbf{R}^m$ be a semialgebraic mapping.

Definition 5. A mapping $\psi: \mathbf{Q}^l \to \mathbf{Q}^l$ is called a local C^k-resolution of u if:

(1) ψ is k-rep.

(2) There is $p = 0, 1, \ldots$, and semialgebraic mappings $\xi_i: \mathbf{R}^l \to \mathbf{R}^m$, $i = 1, \ldots, p$, which are analytic in some neighborhood of $\mathbf{Q}^l$ and satisfy $\| d^s \xi_i \|_{\mathbf{Q}^l} \leqq 1$, $s = 1, \ldots, k$, such that for any $x \in \mathbf{Q}^l$, $\xi_i(x) \neq \xi_j(x)$, $i \neq j$, and $\pi'[(\psi(x) \times \mathbf{R}^m) \cap \Gamma_u] = \{\xi_1(x), \ldots, \xi_p(x)\}$.

If $p = 0$, $\psi(\mathbf{Q}^l) \cap D_u = \varnothing$.

For u univalent, this definition means simply that $\xi = u \circ \psi$ is analytic and satisfies $\| d^s \xi \|_{\mathbf{Q}^l} \leqq 1$, $s = 1, \ldots, k$.

Definition 6. Let $u: \mathbf{Q}^l \to \mathbf{R}^m$ be a semialgebraic mapping. Let $\rho > 0$ be given.

A C^k_ρ-resolution Ψ of u is a finite collection of mappings $\psi_i: \mathbf{Q}^l_\rho \to \mathbf{Q}^l$, or $\psi_i: \mathbf{Q}^l \to \mathbf{Q}^l$, such that $\bigcup_i \operatorname{Im} \psi_i = \mathbf{Q}^l$, and each ψ_i is either a (ρ, k)-reparametrization or a local C^k-resolution of u.

If each ψ_i satisfies (*), Ψ is called a $C^k_{\rho,*}$-resolution of u.

$\kappa(\Psi)$ is defined as the number of mappings ψ_i in Ψ and $d(\Psi)$ is defined as (one of) diagrams d, such that any semialgebraic function ψ_i, $i = 1, \ldots, \kappa(\Psi)$, can be represented with the diagram d.

The main result of this paper is the following:

THEOREM 1. *For any semialgebraic mapping* $u: \mathbf{Q}^l \to \mathbf{R}^m$, $\|u\| \leqq 1$, *natural* k *and* $\rho > 0$, *there exists a* $C^k_{\rho,*}$-*resolution* Ψ *of* u, *with* $d(\psi)$ *depending only on* $d(u)$ *and* k, *and* $\kappa(\Psi) \leqq \mu |\log \rho|^\nu$, *where the constants* μ *and* ν *depend only on* $d(u)$ *and* k.

COROLLARY 2. *Let* $A \subseteq \mathbf{Q}^l$ *be a semialgebraic set. Let a natural* k *and* $\rho > 0$ *be given. Then there exists a semialgebraic* $A_i \subseteq A$, $i = 1, \dots, \kappa$, $A = \bigcup_i A_i$, *such that for each* i *either*

$$A_i = \mathrm{Im}(\psi_i), \qquad \text{where } \psi_i: \mathbf{Q}^l \to \mathbf{Q}^l \text{ is a } (k, *)\text{-reparametrization,}$$

or

$$A_i \subseteq \mathrm{Im}(\psi_i), \qquad \text{where } \psi_i: \mathbf{Q}^l_\rho \to \mathbf{Q}^l \text{ is a } (\rho, k, *)\text{-reparametrization,}$$

and $\kappa \leqq \mu |\log \rho|^\nu$, *with* μ *and* ν *depending only on* $d(A)$ *and* k.

PROOF OF COROLLARY 2. We apply Theorem 1 to the semialgebraic function $u = \chi_A$.

To prove Theorem 1.4 we need another consequence of Theorem 1:

DEFINITION 7. A C^k-resolution Ψ of a semialgebraic mapping $u: \mathbf{Q}^l \to \mathbf{R}^m$ is a finite collection of mappings $\psi_i: \mathbf{Q}^l \to \mathbf{Q}^l$, such that $\bigcup_i \mathrm{Im}\, \psi_i = D_u$, and each ψ_i is a local C^k-resolution of u. If each ψ_i satisfies $(*)$, Ψ is called a C^k_*-resolution. $\kappa(\Psi)$ and $d(\Psi)$ are defined as above.

THEOREM 3. *Let* $u: \mathbf{Q}^l \to \mathbf{R}^m$ *be a semialgebraic mapping, analytic in some neighborhood of* $\mathbf{Q}^l$. *Assume that* $\|d^s u\|_{\mathbf{Q}^l} \leqq M$, $s = 1, \dots, k+1$.

Then for any $0 < c_1 < c_2$ *there exists a semialgebraic set* $A \subseteq \mathbf{Q}^l$, *satisfying* $\{\|u\| \leqq c_1\} \subseteq A \subseteq \{\|u\| \leqq c_2\}$, *such that for the restriction* $u' = u/A$ *of* u *on the set* A *there is a* C^k_*-*resolution* Ψ, *with* $d(\Psi)$ *depending only on* $d(u)$ *and* k, *and* $\kappa(\Psi) \leqq \mu(\log M)^\nu$, *where the constants* μ *and* ν *depend only on* $d(u)$, k, c_1 *and* c_2.

Assuming Theorem 1 we now prove Theorem 3.

PROOF OF THEOREM 3. Induction on l. Assume that Theorem 3 is proved for $l-1$.

Consider $S = \{\|u\| \leqq c_1\}$ and let $\bar{u} = u/S$. $d(\bar{u})$ depends only on $d(u)$.

We apply Theorem 1 to the mapping $\bar{u}$, with the given $k+1$ and $\rho = \min(1/c'M, (c_2 - c_1)/3cM)$, where the constants c and c', depending only on k, l, m, are defined below.

(Notice, that normalizing u in Theorem 1, and then subdividing $\mathbf{Q}^l$ and reparametrizing new cubes linearly, we replace the assumption $\|u\| \leqq 1$ by $\|u\| \leqq c_1$. μ in the expression for κ is then replaced by $\mu \cdot c_1^l$.)

We find therefore a $C_{\rho,*}^{k+1}$-resolution $\bar{\Psi}$ of $\bar{u}$. Those of $\psi_i \in \bar{\Psi}$, which are local resolutions of $\bar{u}$, will form a part of the required resolution Ψ.

Consider some $\psi = \psi_i : \mathbf{Q}_\rho^l \to \mathbf{Q}^l$, which belongs to $\bar{\Psi}$ and is not a local resolution of $\bar{u}$, but a $(\rho, k+1, *)$-reparametrization.

The mapping $\tilde{u} = u \circ \psi : \mathbf{Q}_\rho^l \to \mathbf{R}^m$ is semialgebraic, with $d(\tilde{u})$ depending only on $d(u)$ and k, and analytic in some neighborhood of $\mathbf{Q}_\rho^l$. Applying the formula for the derivatives of a composition (used in [3]), we obtain

$$\|d^s \tilde{u}\| \leqq cM, \quad s = 1, \ldots, k+1, \qquad \text{where } c \text{ depnds only on } k, l \text{ and } m.$$

Now consider the restriction $\hat{u} = \tilde{u}/\{0\} \times \mathbf{Q}^{l-1}$. Using the induction assumption, we apply Theorem 3 to this mapping $\hat{u} : \mathbf{Q}^{l-1} \to \mathbf{R}^m$, with the same $k+1$ and $c_1' = c_1 + \frac{1}{3}(c_2 - c_1)$, $c_2' = c_1 + \frac{2}{3}(c_2 - c_1)$.

We find thus a C_*^k-resolution $\hat{\Psi}$ of $\hat{u}'$ — a restriction of $\hat{u}$ onto the semialgebraic $\hat{A} \subseteq \{0\} \times \mathbf{Q}^{l-1}$ — satisfying $\{\|\hat{u}\| \leqq c_1'\} \subseteq \hat{A} \subseteq \{\|\hat{u}\| \leqq c_2'\}$.

Let $\hat{\psi} \in \hat{\Psi}$, $\hat{\psi} : \mathbf{Q}^{l-1} \to \{0\} \times \mathbf{Q}^{l-1}$. Consider $\hat{\psi}' : \mathbf{Q}_\rho^l \to \mathbf{Q}_\rho^l$, $\hat{\psi}' = \mathrm{Id}_{[0,\rho]} \times \hat{\psi}$.

Let us show first of all that $\hat{\psi}'$ satisfies condition (*). It is evident, if $\mathbf{Q}_\rho^l$ is represented, as it is written above, as $[0, \rho] \times \mathbf{Q}^{l-1}$.

Consider a general situation $\mathbf{Q}_\rho^l = [0,1]^q \times [0,\rho] \times [0,1]^{l-q-1}$. Then $\hat{\Psi}$ is a C_*^k-resolution of $\hat{u} = \tilde{u}/\mathbf{Q}^q \times \{0\} \times \mathbf{Q}^{l-q-1}$. This means that $\hat{\psi} \in \hat{\Psi}$ satisfies condition (*), i.e.

$$\begin{aligned} &\hat{\psi}(x_1, \ldots, x_q, x_{q+2}, \ldots, x_l) \\ &\quad = (\hat{\psi}_1(x_1, \ldots, x_l), \hat{\psi}_2(x_2, \ldots, x_l), \ldots, \hat{\psi}_q(x_q, x_{q+2}, \ldots, x_l), \hat{\psi}_{q+2}(x_{q+2}, \ldots, x_l), \ldots). \end{aligned}$$

But then

$$\begin{aligned} &\hat{\psi}'(x_1, \ldots, x_q, x_{q+1}, x_{q+2}, \ldots, x_l) \\ &\quad = (\hat{\psi}_1(x_1, \ldots, x_l), \ldots, \hat{\psi}_q(x_q, x_{q+2}, \ldots, x_l), x_{q+1}, \hat{\psi}_{q+2}(x_{q+2}, \ldots, x_l), \ldots, \hat{\psi}_l(x_l)) \end{aligned}$$

also satisfies (*).

Clearly, $\|d^s \hat{\psi}'\| \leqq 1$, $s = 1, \ldots, k$. Hence, as above, $\|d^s \hat{u}\| \leqq c'M$, $s = 1, \ldots, k$, with c' depending only on k, l, m, where $\hat{u} = \tilde{u} \circ \hat{\psi}'$.

Moreover, the same inequality is true for any partial derivative of $\hat{u}$ of order $k+1$, containing at least one differentiation with respect to the first coordinate x_1 in $\mathbf{Q}_\rho^l$. Indeed, by the special form of $\hat{\psi}'$,

$$\frac{\partial}{\partial x_1} \hat{u} = \frac{\partial \tilde{u}}{\partial x_1} \circ \hat{\psi}'.$$

Then, differentiating this expression k times, we obtain the formula, containing only the derivatives of $\hat{\psi}'$ up to order k, which are bounded by 1, while the derivatives of $\tilde{u}$ up to $k+1$ are bounded by cM, by the inequality above.

Finally, since $\hat{\psi}$ belongs to the C_*^k-resolution of $\hat{u}'$, we have

$$\|\bar{d}^s(\hat{u}/\{0\}\times \mathbf{Q}^{l-1})\| \leqq 1, \quad s=1,\dots,k, \qquad \text{where } \bar{d}=\left(\frac{\partial}{\partial x_2},\dots,\frac{\partial}{\partial x_l}\right).$$

But, as was shown above,

$$\left\|\frac{d}{dx_1}(\bar{d}^s\hat{u})\right\| \leqq c'M, \qquad s=1,\dots,k,$$

and since the size ρ of $\mathbf{Q}^l_\rho$ in the direction x_1 satisfies $\rho \leqq 1/c'M$, we get $\|\bar{d}^s\hat{u}\| \leqq 2$ on $\mathbf{Q}^l_\rho$, $s=1,\dots,k$.

Subdividing $\mathbf{Q}^{l-1}$, we can assume $\|\bar{d}^s\hat{u}\| \leqq 1$, $s=1,\dots,k$.

Now let $\alpha: \mathbf{Q}^l \to \mathbf{Q}^l_\rho$ be the mapping

$$(x_1,x_2,\dots,x_l)\to(\rho x_1,x_2,\dots,x_l).$$

For $\hat{u}\circ\alpha$, all the derivatives $\bar{d}$ remain the same as for $\hat{u}$, but any derivative of $\hat{u}$ containing differentiations with respect to x_1, is multiplied by the positive power of ρ.

Since $\|d^s\hat{u}\| \leqq c'M$, $s=1,\dots,k$, and $\rho \leqq 1/c'M$, we get

$$\|d^s(\hat{u}\circ\alpha)\| < 1, \qquad s=1,\dots,k.$$

Now, $\hat{u}\circ\alpha=\tilde{u}\circ\hat{\psi}'\circ\alpha=u\circ\psi\circ\hat{\psi}'\circ\alpha$, with $\psi''=\psi\circ\hat{\psi}'\circ\alpha: \mathbf{Q}^l\to\mathbf{Q}^l$ semialgebraic, analytic and diffeomorphic in some neighborhood of $\mathbf{Q}^l$, $d(\psi'')$ depends only on $d(\psi)$ and $d(\hat{\psi}')$, and hence only on $d(u)$ and k.

Since all the derivatives up to order k of ψ and $\hat{\psi}'$ are bounded by 1, and α is a linear contraction,

$$\|d^s\psi''\| < c''(k,l), \qquad s=1,\dots,k.$$

Subdividing $\mathbf{Q}^l$ into the subcubes of the size $1/c''$ and reparametrizing linearly, we obtain new mappings (which we denote also by ψ''), satisfying

$$\|d^s\psi''\| \leqq 1, \qquad s=1,\dots,k.$$

It was shown above that

$$\|d^s(u\circ\psi'')\| \leqq 1, \qquad s=1,\dots,k.$$

Hence, the mappings ψ'' are local C_*^k-resolutions of u. (Since ψ'' is a composition of mappings, satisfying (*), it also satisfies (*).)

Now we define the required resolution Ψ as the collection of all the mappings $\psi \in \Psi$, which are local resolutions of $\bar{u}$, with $\operatorname{Im} \psi_i \subseteq D\bar{u}$, and all the mapping ψ'', built above for those $\psi_i \in \bar{\Psi}$, which are not local resolutions of $\bar{u}$.

The set $A \subseteq \mathbf{Q}^l$ we define as the union of the images of all the $\psi \in \Psi$.

Clearly, Ψ is a C_*^k-resolution of $u' = u/A$.

It remains to show that

$$\{\|u\| \leqq c_1\} \subseteq A \subseteq \{\|u\| \leqq c_2\},$$

and to count the mappings $\psi \in \Psi$.

Let us prove first that $S = \{\|u\| \leqq c_1\} \subseteq A$. Let $x \in S$. Since $\bar{\Psi}$, built above, is a $C_{\rho,*}^{k+1}$-resolution of $\bar{u}/S$, the images of $\psi_i \in \bar{\Psi}$ cover $\mathbf{Q}^l$. If $x \in \operatorname{Im} \psi_i$ for ψ_i a local resolution of $\bar{u}$, then $\operatorname{Im} \psi_i \cap S \neq \varnothing$ and then $\operatorname{Im} \psi_i \subseteq S$. Thus by construction, $\psi_i \in \Psi$, and hence $x \in \operatorname{Im} \psi_i \subseteq A$.

Let $x \in \operatorname{Im} \psi_i$ with $\psi_i \in \bar{\Psi}$ a $(\rho, k+1, *)$-reparametrization. Let $y = \psi_i^{-1}(x)$, $y \in \mathbf{Q}_\rho^l$. $\bar{u}(y) = u \circ \psi_i(y) = u(x)$, and hence $\|\bar{u}(y)\| \leqq c_1$. Let $y = (y_1, \ldots, y_l)$.

Consider the point $y' = (0, y_2, \ldots, y_l) \in \{0\} \times \mathbf{Q}^{l-1}$. Since $\|d\bar{u}\| \leqq cM$, and $\rho \leqq (c_2 - c_1)/3cM$, $\|\bar{u}(y) - \bar{u}(y')\| \leqq \frac{1}{3}(c_2 - c_1)$ and hence $\|\bar{u}(y')\| \leqq c_1 + \frac{1}{3}(c_2 - c_1) = c_1'$; therefore $y' \in D_{\hat{u}'}$.

But then y' is covered by one of the mappings $\hat{\psi}$ belonging to the resolution $\hat{\Psi}$ of $\hat{u}'$, and consequently, $y \in \operatorname{Im} \hat{\psi}'$, where $\hat{\psi}' = \operatorname{Id}_{[0,\rho]} \times \hat{\psi}$, and $x = \psi_i(y)$ is contained in the image of some of $\psi'' = \psi_i \hat{\psi}' \circ \alpha$. Since $\psi'' \in \Psi$, $x \in A$.

Conversely, let $x \in A$. If $x \in \operatorname{Im} \psi_i$, with $\psi_i \in \bar{\Psi}$ a local resolution and $\operatorname{Im} \psi_i \subseteq S$, then $x \in S = \{\|u\| \leqq c_1\}$.

Let $x \in \operatorname{Im} \psi''$. Then $u(x) = \bar{u}(y)$, where $y = \psi_i^{-1}(x) \in \mathbf{Q}_\rho^l$. Let, as above, $y = (y_1, \ldots, y_l)$, $y' = (0, y_2, \ldots, y_l)$. Since $y \in \operatorname{Im} \hat{\psi}'$, $y' \in D_{\hat{u}'}$, i.e. $\|\bar{u}(y')\| \leqq c_2' = c_1 + \frac{2}{3}(c_2 - c_1)$. But, as above,

$$\|\bar{u}(y) - \bar{u}(y')\| \leqq \tfrac{1}{3}(c_2 - c_1),$$

and we obtain

$$\|u(x)\| = \|\bar{u}(y)\| \leqq c_1 + \tfrac{2}{3}(c_2 - c_1) + \tfrac{1}{3}(c_2 - c_1) = c_2.$$

Hence $x \in \{\|u\| \leqq c_2\}$, and $A \subseteq \{\|u\| \leqq c_2\}$.

To bound the number of mappings in Ψ we notice that by Theorem 1, the number of mappings in $\bar{\Psi}$ is at most

$$\kappa_1 = c_1^l \mu \,|\log \rho|^\nu \leqq c_1^l \mu_1 \cdot \tilde{c}^{\nu_1} (\log M)^{\nu_1},$$

where μ_1 and ν_1 depend only on $d(u)$ and k, and $\tilde{c}$ depends on k, l, m, c_1 and c_2.

Some of these mappings we reparametrize, using an inductive assumption for Theorem 3, by at most κ_2 new mappings, with

$$\kappa_2 = \mu_2(\log cM)^{\nu_2} = \hat{c}\mu_2(\log M)^{\nu_2},$$

where $\hat{c}$ depends only on k, l, m (we assume $M \geqq 1$), μ_2 and ν_2 depend on $d(\tilde{u})$, k, c'_1, c'_2, i.e. only on $d(u)$, k, c_1, c_2.

Some of the new mappings we reparametrize linearly by at most 2^{l-1} new mappings, and then once more by $(c'')^l$ new mappings. Thus

$$\begin{aligned}\kappa(\Psi) &\leqq \kappa_1 \cdot \kappa_2 \cdot 2^{l-1} \cdot (c'')^l \\ &\leqq c'^l_1 \mu_1 \bar{c}^{\nu_1}(\log M)^{\nu_1} \cdot \hat{c}\mu_2(\log M)^{\nu_2} \cdot 2^{l-1}(c'')^l \\ &= \mu(\log M)^{\nu},\end{aligned}$$

with μ and ν depending only on $d(u)$, k, c_1 and c_2.

Theorem 3 is proved.

Proof of Theorem 1. We use induction on l. For $l = 0$ the statement is evident. So let us assume that for $l - 1 \geqq 0$ Theorem 1 is proved.

Let $u: \mathbf{Q}^l \to \mathbf{R}^m$ be a semialgebraic mapping, $\|u\| \leqq 1$, and let a natural k and $\rho > 0$ be given.

We have to find a $C^k_{\rho,*}$-resolution Ψ of u, with $d(\Psi)$ depending only on $d(u)$ and k and $\kappa(\Psi) \leqq \mu\,|\log \rho\,|^{\nu}$.

First we show that it is enough to prove a weaker version of this theorem.

Definition 8. $\psi: \mathbf{Q}^l \to \mathbf{Q}^l$ is called a local $C^{k,q}$-resolution of u, $q = 1, \dots, k$, if it satisfies the condition (1) of Definition 5, and condition (2), with the inequality $\|d^s\xi_i\|_{\mathbf{Q}^l} \leqq 1$, $s = 1, \dots, k$, replaced by $\|d^s\xi_i\|_{\mathbf{Q}^l} \leqq 1$, $s = 1, \dots, q$.

A $C^{k,q}_\rho$-resolution of u is defined respectively, as well as a $C^{k,q}_{\rho,*}$-resolution.

$C^{k,k}$-resolutions then coincide with C^k-resolutions, defined above.

Lemma 4. *If for any u as above, k, ρ, there exists a $C^{k,1}_\rho$-resolution with the required properties, then there exists also a C^k_ρ-resolution.*

Proof. Induction by q. Let Ψ be a $C^{k,q}_\rho$-resolution of u. Those of $\psi \in \Psi$, which are (ρ, k)-reparametrization, will appear also in the required $C^{k,q+1}_\rho$-resolution. So let $\psi \in \Psi$, $\psi: \mathbf{Q}^l \to \mathbf{Q}^l$, be a local $C^{k,q}$-resolution of u.

The composition $u \circ \psi$ is represented by p semialgebraic mappings $\xi_i: \mathbf{Q}^l \to \mathbf{R}^m$, $i = 1, \dots, p$, which are analytic in the neighborhood of $\mathbf{Q}^l$ and satisfy $\|d^s\xi_i\|_{\mathbf{Q}^l} \leqq 1$, $s = 1, \dots, q$.

Consider the semialgebraic mapping

$$w: \mathbf{Q}^l \to W \times \overbrace{W \times \cdots \times W}^{p}, \ w(x) = (d^q\xi_1(x), \ldots, d^q\xi_p(x)),$$

where W is the space of q-polylinear mappings $\mathbf{R}^l \to \mathbf{R}^m$, and the norm in $W \times \cdots \times W$ is defined as the maximum of norms in W. Let $\bar{\Psi}$ be the $C_\rho^{k,1}$-resolution of w. (By assumption, $\|w\| \leqq 1$.) If $\bar{\psi} \in \bar{\Psi}$ is a (ρ, k)-reparametrization, then also $\psi \circ \bar{\psi}$ is a (ρ, k)-reparametrization (maybe after subdividing Q^l into a number of smaller subcubes, depending only on k and l, and linear reparametrizations).

So assume that $\bar{\psi} \in \bar{\Psi}$ is a local $C^{k,1}$-resolution of w. This means, in particular, that $\|\bar{d}(w \circ \bar{\psi})\| \leqq 1$, where $\bar{d}$ denotes the differentiation with respect to the "new coordinates". By definition of w we obtain

$$\|\bar{d}((d^q\xi_i) \circ \psi)\| \leqq 1, \qquad i = 1, \ldots, p.$$

Now we show that $\|\bar{d}^s(\xi_i \circ \bar{\psi})\|$, $s = 1, \ldots, q+1$, are bounded by constants, depending only on k, l, m.

For $s = 1, \ldots, q$ this follows immediately from the fact that $\|d^s\xi_i\| \leqq 1$, $s = 1, \ldots, q$, and $\|\bar{d}^s\bar{\psi}\| \leqq 1$, $s = 1, \ldots, k$, since $\bar{\psi} \in \bar{\Psi}$. Now,

$$\bar{d}^q(\xi_i \circ \bar{\psi}) = \sum_{j=1}^{q} d^j\xi_i(\bar{\psi}) \circ p_j(\bar{d}\bar{\psi}),$$

where p_j are the standard polynomials in derivatives of $\bar{\psi}$ up to order q. Hence

$$\bar{d}^{q+1}(\xi_i \circ \bar{\psi}) = \bar{d}(\bar{d}^q(\xi_i \circ \bar{\psi})) = \sum_{j=1}^{q} \bar{d}(d^j\xi_i(\bar{\psi})) \circ p_j(\bar{d}\psi) + d^j\xi_i(\bar{\psi}) \circ \bar{d}(p_j(\bar{d}\bar{\psi})).$$

The second term for each $j = 1, \ldots, q$ is bounded by a constant, depending only on k, l, m, since

$$\|d^j\xi_i(\bar{\psi})\| \leqq 1, \qquad j = 1, \ldots, q,$$

by assumptions, and

$$\|\bar{d}^s\bar{\psi}\| \leqq 1 \qquad \text{for } s = 1, \ldots, q+1 \leqq k.$$

The first term for $j \leqq q-1$ has the form $d^{j+1}\xi_i(\bar{\psi}) \circ \bar{d}\bar{\psi} \circ p_j(\bar{d}(\bar{\psi})$, and hence is bounded by the constant of the same type. For $j = q$ we have $\bar{d}(d^q\xi_i(\bar{\psi})) \circ p_q(\bar{d}\bar{\psi})$, but $\|\bar{d}(d^q\xi_i(\bar{\psi}))\| \leqq 1$, as was shown above.

Thus $\|\bar{d}^s(\xi_i \circ \bar{\psi})\| \leqq C$, $s = 1, \ldots, q+1$, $i = 1, \ldots, p$, with C depending only on k, l, m. Subdividing $\mathbf{Q}^l$ into the subcubes of the size $1/C$ and reparametrizing linearly, we obtain

$$\|\bar{d}^s(\xi_i \circ \bar{\psi})\| \leqq 1, \qquad s = 1, \dots, q+1.$$

Hence $\psi \circ \bar{\psi}$: $\mathbf{Q}^l \to \mathbf{Q}^l$, with mappings $\xi'_i = \xi_i \circ \bar{\psi} = u \circ (\psi \circ \bar{\psi})$, $i = 1, \dots, p$, is a local $C^{k,q+1}$-resolution of u.

The diagram of any of the reparametrizations constructed depends only on $d(u)$ and k (here we use the fact that the diagram of the derivatives of a semialgebraic mapping depends only on its diagram).

The number of reparametrizations is at most

$$\tilde{c} \cdot \kappa(\Psi) \cdot \kappa(\bar{\Psi}) = \tilde{c} \cdot \mu_1 |\log \rho|^{\nu_1} \cdot \mu_2 |\log \rho|^{\nu_2} = \tilde{c} \mu_1 \mu_2 |\log \rho|^{\nu_1 + \nu_2},$$

where $\tilde{c}$ depends only on k, l, m, and μ_1, μ_2, ν_1, ν_2 depend only on $d(u)$ and k. Notice also that the property (*) is preserved in our construction.

Lemma 4 is proved.

It remains to prove the existence of $C^{k,1}_\rho$-resolutions.

Let $u: \mathbf{Q}^l \to \mathbf{R}^m$ be a semialgebraic mapping given by an l-dimensional semialgebraic subset $\Gamma_u \subseteq \mathbf{Q}^l \times \mathbf{R}^m$.

Let $\Sigma_u \subseteq \Gamma_u$ be the subset, consisting of all the singular points of Γ_u. (In the case of real algebraic and semialgebraic sets the definition of singular points involves some difficulties; we can use, for example, the following construction: there exists a stratification of a given semialgebraic set Γ_u, with a diagram, depending only on $d(u)$. Then we take as Σ_u the union of all the strata of dimensions $\leqq l-1$ in this stratification. (The same remark concerns the set Σ'_u below.)

Thus Σ_u is a closed semialgebraic set in $\mathbf{R}^l \times \mathbf{R}^m$, of dimension at most $l-1$.

Let Σ'_u be the set of critical points of the projection π/Γ_u, where π: $\mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^m$. The dimension of Σ'_u can be l, but $\dim \pi(\Sigma'_u) \leqq l-1$. Thus $\Delta_u = \pi(\Sigma_u \cup \Sigma'_u)$ is a closed semialgebraic subset in $\mathbf{Q}^l$ of dimension at most $l-1$.

Finally denote by Δ the set $\Delta_u \cup \partial D_u \cup \partial \mathbf{Q}^l$. $d(\Delta)$ depends only on $d(u)$ and $\dim \Delta = l-1$.

By construction of Δ, over each connected component S_i of $\mathbf{Q}^l \setminus \Delta$, u is represented by p_i analytic mappings

$$\eta^i_j\colon S_i \to \mathbf{R}^m, \quad j = 1, \dots, p_i, \quad \eta^i_{j_1}(x) \neq \eta^i_{j_2}(x) \quad \text{for each } x \in S_i, \quad j_1 \neq j_2.$$

Now $\Delta \subseteq \mathbf{Q}^l = [0,1] \times \mathbf{Q}^{l-1}$ is a closed semialgebraic subset of dimension $l-1$, so Δ can be considered as the graph Γ_δ of the semialgebraic function $\delta: \mathbf{Q}^{l-1} \to \mathbf{R}$, $x_1 = \delta(x_2, \dots, x_l)$, where $x_1, \dots, x_l$ are the coordinates in $\mathbf{Q}^l$. We have $\|\delta\| = 1$.

Using the induction assumption, we apply Theorem 1 to the function δ of $l-1$ variables. Let Ψ' be the $C^k_{\rho,*}$-resolution of δ.

Let $\psi' \in \Psi'$ be a $(\rho, k, *)$-reparametrization. ψ' is a mapping of $\mathbf{Q}^{l-1}_\rho \to \{0\} \times \mathbf{Q}^{l-1}$. Then the mapping

$$\psi = \mathrm{Id}_{[0,1]} \times \psi' : [0,1] \times \mathbf{Q}^{l-1}_\rho \to [0,1] \times \mathbf{Q}^{l-1}$$

is a $(\rho, k, *)$-reparametrization. All the mappings ψ, obtained in this way, will form a part of the required resolution Ψ.

Now let $\psi' \in \Psi'$ be a local resolution of δ, $\psi' : \mathbf{Q}^{l-1} \to \{0\} \times \mathbf{Q}^{l-1}$. By definition, $\Delta \cap ([0,1] \times \mathrm{Im}\, \psi')$ consists of r hypersurfaces

$$x_1 = \alpha_1(x_2, \ldots, x_l), x_1 = \alpha_2(x_2, \ldots, x_l), \ldots, x_1 = \alpha_r(x_2, \ldots, x_l).$$

$\alpha_1 < \alpha_2 < \cdots < \alpha_r$ for any $x' = (x_2, \ldots, x_l) \in \mathrm{Im}\, \psi'$, and $\alpha_i \circ \psi'$ satisfy $\| d^s(\alpha_i \circ \psi') \| \leqq 1$, $i = 1, \ldots, r$, $s = 1, \ldots, k$.

Fix some $i = 1, \ldots, r-1$, and define $\psi_i : \mathbf{Q}^l \to \mathbf{Q}^l$ by

$$\psi_i(t, x') = (\alpha_i(\psi'(x')) + t(\alpha_{i+1}(\psi'(x')) - \alpha_i(\psi'(x'))), \psi'(x')),$$

where $x' \in Q^{l-1}$.

Clearly, ψ_i is an analytic diffeomorphism of some neighborhood of $\mathbf{Q}^l$, and since ψ' satisfies $(*)$, the expression for ψ_i shows that it also satisfies $(*)$.

The diagram $d(\psi_i)$ depends only on the diagrams $d(\delta)$ and $d(\psi')$, and hence, only on $d(u)$ and k.

Finally, since $\| d^s(\alpha_i \circ \psi') \| \leqq 1$ and $\| d^s \psi' \| \leqq 1$, $s = 1, \ldots, k$, also $\| d^s \psi_i \| \leqq 2$, and subdividing $\mathbf{Q}^l$ into 2^l subcubes and reparametrizing linearly, we can assume that $\| d^s \psi_i \| \leqq 1$, $i = 1, \ldots, k$.

Now from ψ_i we construct three reparametrizations: $\psi'_i = \psi_i / [0, \rho] \times \mathbf{Q}^{l-1}$, $\psi'''_i = \psi_i / [\rho, 1-\rho] \times \mathbf{Q}^{l-1}$, composed with the linear mapping of $\mathbf{Q}^l$ onto $[\rho, 1-\rho] \times \mathbf{Q}^{l-1}$, and $\psi''_i = \psi_i / [1-\rho, 1] \times \mathbf{Q}^{l-1}$.

Thus ψ'_i and ψ''_i are the $(\rho, k, *)$-reparametrizations, and they will also participate in the required resolution Ψ.

Clearly, the images of all the reparametrizations constructed, cover $\mathbf{Q}^l$. It remains to transform the mappings of the form ψ'''_i above into the local $C^{k,1}_{\rho,*}$-resolutions of u.

By construction, $\mathrm{Im}\, \psi'''_i$ is contained in one of connected components of $\mathbf{Q}^l \setminus \Delta$, say S_j. Then $u \circ \psi'''_i$ is represented by p_j analytic mappings $\eta^j_q \circ \psi'''_i$, $q = 1, \ldots, p_j$.

Moreover, transforming subsequently each coordinate function of these mappings, we can reduce the problem to the case of a real semialgebraic function. Thus it is sufficient to prove the following lemma:

LEMMA 5. *Let $u: \mathbf{Q}^l \to \mathbf{R}$ be a semialgebraic function, analytic on some neighborhood of $\mathbf{Q}^l$ and satisfying on $\mathbf{Q}^l$ the inequality $0 \leqq u \leqq 1$. Then for a natural k and $\rho > 0$, there exists a $C^{k,1}_{\rho,*}$-resolution Ψ of u, with $d(\Psi)$ depending only on $d(u)$ and k, and*

$$\kappa(\Psi) \leqq \mu(d(u), k) |\log \rho|^{\nu(d(u),k)}.$$

PROOF. We use induction on the number of variables, with respect to which the first derivatives of u are already small.

DEFINITION 9. $\psi: \mathbf{Q}^l \to \mathbf{Q}^l$ is called a local $C^{k,1,r}_*$-resolution of u, if ψ is $(k,*)$-reparametrization and

$$\left| \frac{\partial(u \circ \psi)}{\partial x_i} \right|_{\mathbf{Q}^l} \leqq 1, \qquad i = 1, \ldots, r.$$

A $C^{k,1,r}_{\rho,*}$-resolution of u is defined respectively.

For $r = l$ we obtain $C^{k,1}$-resolutions.

To start the induction on r, let us prove the existence of $C^{k,1,1}_*$-resolutions.

Let, as above, $x_1, \ldots, x_l$ be the coordinates in $\mathbf{Q}^l$. Consider in $\mathbf{Q}^l$ the semialgebraic subset

$$\Delta = \partial \mathbf{Q}^l \cup \left\{ \frac{\partial u}{\partial x_1} = 0 \right\} \cup \left\{ \frac{\partial^2 u}{\partial x_1^2} = 0 \right\}.$$

(If one or both of the derivatives are identically zero, we do not consider the corresponding equation.)

$d(\Delta)$ depends only on $d(u)$ and $\dim \Delta = l - 1$.

Consider $\Delta \subseteq [0,1] \times \mathbf{Q}^{l-1}$ as the graph Γ_δ of the semialgebraic function $x_1 = \delta(x_2, \ldots, x_l)$.

We apply Theorem 1 to the function δ of $l-1$ variables. Let Ψ' be a $C^k_{\rho,*}$-resolution of δ. Exactly as above we use Ψ' to produce reparametrizations $\psi: \mathbf{Q}^l \to \mathbf{Q}^l$, satisfying $(*)$, whose images cover $\mathbf{Q}^l$. Some of ψ are ρ-small, and others have the form

$$\psi(t, x') = (\alpha(\psi'(x')) + t(\alpha'(\psi'(x')) - \alpha(\psi'(x'))), \psi'(x')), \qquad \alpha' > \alpha,$$

and their images are contained in connected components of $\mathbf{Q}^l \setminus \Delta$.

For any ψ of this form consider the composition $u' = u \circ \psi$. From the expression for ψ it is clear that the partial derivatives of u' with respect to the first coordinate are equal to the corresponding derivatives of u, multiplied by positive constants. Hence $\partial u'/\partial x_1$ and $\partial^2 u'/\partial x_1^2$ do not change signs on all $\mathbf{Q}^l$.

Assume, for example, that $\partial u'/\partial x_1 \geqq 0$, $\partial^2 u'/\partial x_1^2 \geqq 0$. (Other cases are considered exactly in the same way.) Then $\partial u'/\partial x_1$ is nondecreasing on any "vertical" line $x' = (x_2, \ldots, x_l) = \text{const}$.

Consider the subcubes

$$\mathbf{Q}_j^l = [1-(1/2)^{j-1}, 1-(1/2)^j] \times \mathbf{Q}^{l-1}, \qquad j = 1, \ldots, [|\log \rho|] + 1.$$

We claim that for any $x \in \mathbf{Q}_j^l$,

$$\frac{\partial u'}{\partial x_1}(x) \leqq 2^j.$$

Indeed, if for $x = (x_1, x')$, $x_1 \leqq 1-(1/2)^j$, $\partial u'/\partial x_1 > 2^j$, then at any point (y, x'), $y \geqq 1-(1/2)^j$, also $\partial u'/\partial x_1 > 2^j$. Hence

$$u'((1, x')) - u'((1-(1/2)^j, x')) = \int_{1-(1/2)^j}^{1} \frac{\partial u'}{\partial x_i}(y, x')dy > 1,$$

and this contradicts the inequality $0 \leqq u' \leqq 1$.

Now we reparametrize each $\mathbf{Q}_j^l$ by the mapping

$$\beta_j : \mathbf{Q}^l \to \mathbf{Q}_j^l, \quad \beta_j(t, x') = (1-(1/2)^{j-1} + (1/2)^j \cdot t, x').$$

Since

$$\frac{\partial}{\partial t}(u' \circ \beta_j) = \frac{\partial u'}{\partial x_1} \cdot (1/2)^j, \quad \text{and} \quad \frac{\partial u'}{\partial x_1} \leqq 2^j \quad \text{on } \mathbf{Q}_j^l,$$

we get

$$|\partial(u' \circ \beta_j)/\partial x_1| \leqq 1 \quad \text{on } \mathbf{Q}^l.$$

The part of $\mathbf{Q}^l$, which is not covered by $\mathbf{Q}_j^l$, has the form $[1-\rho', 1] \times \mathbf{Q}^{l-1}$, where $\rho' = (1/2)^{[|\log \rho'|]+1} \leqq \rho$. Hence this part is the image of a (linear) $(\rho, k, *)$-reparametrization.

Thus we constructed a $C_{\rho,*}^{k,1,1}$-resolution of u, with the required diagram and number of reparametrizations.

Notice that just in this construction the necessity of ρ-small reparametrizations and the number $|\log \rho|$ of partitions appear explicitly, as well as the using of the condition $\|u\| \leqq 1$. In the rest of the proof these points are somewhat obscure, because the induction arguments are used.

So let us assume that the required $C_{\rho,*}^{k,1,r}$-resolution of u is constructed, $r \geqq 1$. We want to transform it to $C_{\rho,*}^{k,1,r+1}$-resolution by further subdivision of its local resolutions. Considering one of these local resolutions, we can assume that $u : \mathbf{Q}^l \to \mathbf{R}$, $0 \leqq u \leqq 1$, satisfies $|\partial u/\partial x_i|_{\mathbf{Q}^l} \leqq 1$, $i = 1, \ldots, r$.

Define $\Delta \subseteq \mathbf{Q}^l$ as

$$\partial \mathbf{Q}^l \cup \left\{ \frac{\partial u}{\partial x_{r+1}} = 0 \right\} \cup \left\{ \frac{\partial^2 u}{\partial x_1 \partial x_{r+1}} = 0 \right\},$$

and let $\delta(x_2, \ldots, x_l)$ be the function with the graph $\Gamma_\delta = \Delta$.

Now we apply Theorem 1 to two functions of $l-1$ variables simultaneously: δ and u/Δ (they can be considered as the coordinates of the semialgebraic mapping to the plane). Let Ψ' be a corresponding $C^k_{\rho,*}$-resolution and let $\psi' \in \Psi'$ be a local resolution. Thus Δ over $\operatorname{Im} \psi'$ in $\{0\} \times \mathbf{Q}^{l-1}$ consists of p hypersurfaces

$$x_1 = \alpha_1(x_2, \ldots, x_l), \ldots, x_1 = \alpha_p(x_2, \ldots, x_l), \qquad \alpha_1 < \alpha_2 < \cdots < \alpha_p,$$

and

$$\| d^s(\alpha_i \circ \psi') \| \leqq 1 \quad \text{and} \quad \| d^s(u(\alpha_i \circ \psi', \psi')) \| \leqq 1, \qquad s = 1, \ldots, k, \quad i = 1, \ldots, p.$$

Denote the coordinates in the source of $\psi' : \mathbf{Q}^{l-1} \to \{0\} \times \mathbf{Q}^{l-1}$ by $v_2, \ldots, v_l$, and rewrite the above inequalities for $s = 1$ in a more explicit form:

$$\left| \frac{d}{dv_j} (\alpha_i \circ \psi') \right| \leqq 1, \qquad j = 2, \ldots, l,$$

and

$$\left| \frac{\partial u}{\partial x_1} \cdot \frac{d}{dv_j} (\alpha_i \circ \psi') + \sum_{q=2}^{l} \frac{\partial u}{\partial x_q} \cdot \frac{\partial \psi'_q}{\partial v_j} \right| \leqq 1, \qquad j = 2, \ldots, l,$$

where $\psi'(v) = (0, \psi'_2, \ldots, \psi'_l)$.

Since ψ' satisfies condition (*) we have $\partial \psi'_q / \partial v_j \equiv 0$ for $j < q$. Thus, for $j = r+1$, the inequalities above take the form

$$\text{(1)} \qquad \begin{aligned} &\left| \frac{d}{dv_{r+1}} (\alpha_i \circ \psi') \right| \leqq 1, \\ &\left| \frac{\partial u}{\partial x_1} \cdot \frac{d}{dv_{r+1}} (\alpha_i \circ \psi') + \sum_{q=2}^{r+1} \frac{\partial u}{\partial x_q} \cdot \frac{\partial \psi'_q}{\partial v_{r+1}} \right| \leqq 1. \end{aligned}$$

Now, since $r \geqq 1$, $|\partial u / \partial x_1|_{\mathbf{Q}^l} \leqq 1$, as well as $|\partial u / \partial x_q|_{\mathbf{Q}^l}$, for $q \leqq r$. Also $|\partial \psi'_q / \partial v_{r+1}| \leqq 1$, $q = 2, \ldots, r+1$, since ψ' is a C^k-reparametrization.

Hence all the terms in (1), except the last one, are bounded by 1, and since all the expression (1) is bounded by 1, we obtain

$$\left| \frac{\partial u}{\partial x_{r+1}} ((\alpha_i \circ \psi', \psi')) \cdot \frac{\partial \psi_{r+1}}{\partial v_{r+1}} \right| \leqq r+1 \qquad \text{on } \mathbf{Q}^{l-1},$$

and this inequality is satisfied on each hypersurface α_i, $i = 1, \ldots, p$.

Now we fix some $i = 1, \dots, p-1$, and reparametrize the part of $\mathbf{Q}^l$, lying over $\operatorname{Im}\psi'$ between the hypersurfaces α_i and α_{i+1}, by $\psi : \mathbf{Q}^l \to \mathbf{Q}^l$,

$$\psi(t, v) = (\alpha_i \circ \psi'(v) + t(\alpha_{i+1} \circ \psi'(v) - \alpha_i \circ \psi'(v)), \psi'(v)).$$

We prove that for $u' = u \circ \psi$,

$$\left|\frac{\partial u'}{\partial t}\right| \leqq c \quad \text{and} \quad \left|\frac{\partial u'}{\partial v_j}\right| \leqq c, \qquad j = 2, \dots, r+1, \quad \text{on } \mathbf{Q}^l,$$

where c depends only on r.

For $\partial u'/\partial t$ we have

$$\frac{\partial u'}{\partial t} = \frac{\partial u}{\partial x_1} \cdot (\alpha_{i+1} - \alpha_i) \leqq 1.$$

For $j = 2, \dots, r$

$$\text{(2)} \quad \frac{\partial u'}{\partial v_j} = \frac{\partial u}{\partial x_1} \cdot \left(\frac{\partial}{\partial v_j}(\alpha_i \circ \psi') + t\left(\frac{\partial}{\partial v_j}(\alpha_{i+1} \circ \psi') - \frac{\partial}{\partial v_j}(\alpha_i \circ \psi')\right)\right) + \sum_{q=2}^{j} \frac{\partial u}{\partial x_q} \cdot \frac{\partial \psi'_q}{\partial v_j},$$

and since all the terms in this expression are bounded by 1, we get $|\partial u'/\partial v_j| \leqq c(r)$ on $\mathbf{Q}^l$.

Finally, for $\partial u'/\partial v_{r+1}$ we obtain

$$\frac{\partial u'}{\partial v_{r+1}} = (A) + \frac{\partial u}{\partial x_{r+1}} \cdot \frac{\partial \psi'_{r+1}}{\partial v_{r+1}},$$

where the expression (A) contains exactly the same terms as (2) with v_j replaced by v_{r+1}. All these terms are bounded by 1, therefore $|(A)| \leqq c(r)$. It remains to estimate the last term.

Each point of the image of ψ has the form $(x_1, \psi'(v))$, $v \in \mathbf{Q}^{l-1}$, where $\alpha_i \circ \psi'(v) \leqq x_1 \leqq \alpha_{i+1} \circ \psi'(v)$.

Now by construction, the interior of the $\operatorname{Im}\psi$ is contained in $\mathbf{Q}^l \setminus \Delta$, i.e. $\partial u/\partial x_{r+1}$ and $\partial^2 u/\partial x_1 \partial x_{r+1}$ preserve sign there. In particular, $\partial u/\partial x_{r+1}$ is monotone on each "vertical" line $x' = (x_2, \dots, x_l) = \text{const}$, inside the $\operatorname{Im}\psi$. Hence the following inequality is satisfied:

$$\left|\frac{\partial u}{\partial x_{r+1}}((x_1, \psi'(v)))\right|$$
$$\leqq \max\left(\left|\frac{\partial u}{\partial x_{r+1}}((\alpha_i \circ \psi'(v), \psi'(v)))\right|, \left|\frac{\partial u}{\partial x_{r+1}}((\alpha_{i+1} \circ \psi'(v), \psi'(v)))\right|\right),$$

for any x_1 satisfying $\alpha_i \circ (v) \leqq x_1 \leqq \alpha_{i+1} \circ \psi'(v)$.

But we have shown above that on hypersurfaces α_j the inequality

$$\left|\frac{\partial u}{\partial x_{r+1}}(\alpha_j \circ \psi', \psi')\right| \cdot \left|\frac{\partial \psi'_{r+1}}{\partial v_{r+1}}\right| \leqq r+1$$

is satisfied. Hence at any point in $\operatorname{Im} \psi$,

$$\left|\frac{\partial u}{\partial x_{r+1}} \cdot \frac{\partial \psi'_{r+1}}{\partial v_{r+1}}\right| \leqq r+1,$$

and

$$\left|\frac{du'}{dv_{r+1}}\right| \leqq c'(r) \qquad \text{on } \mathbf{Q}'.$$

By the standard additional subdivision we can get

$$\left|\frac{\partial u'}{\partial v_j}\right| \leqq 1, \qquad j = 1, \ldots, r+1.$$

This completes the proof of Lemma 5 and Theorem 1.

Now we complete the proof of theorem 2.1 of [3], and hence also the proof of the main theorem 1.4.

The first two steps in the proof, given in [3], remain the same. Note only that in fact, for the Taylor polynomial p of g, built there, we have

$$\| g - p \|_{C^k} \leqq c(k, l, m),$$

and not only $\| g - p \|_{C^0}$, as was used in [3]. In particular we get

$$\| d^s p \| \leqq c'M, \qquad s = 1, \ldots, k+1.$$

Now at step 3, instead of working with the semialgebraic set $S' = \{\| p \| \leqq \frac{5}{4}\}$, we apply Theorem 3 to the polynomial p. We find a C^k-resolution Ψ of p/A, where $S' \subseteq A$, and $\kappa(\Psi) \leqq \mu(\log M)^\nu$.

Let $\psi \in \Psi$. By definition of Ψ,

$$\| d^s(p \circ \psi) \| \leqq 1, \qquad s = 1, \ldots, k.$$

But then the inequality $\| g - p \|_{C^k} \leqq c$ implies

$$\| d^s(g \circ \psi) \| \leqq c'', \qquad s = 1, \ldots, k.$$

Indeed,

$$\| d^s(p \circ \psi) - d^s(g \circ \psi) \| \leqq \sum_{j=1}^{s} \| d^j p(\psi) - d^j g(\psi) \| \cdot \| p_j(d\psi) \| \leqq c \cdot c''' = c'',$$

where c''' is the bound for $\| p_j(d\psi) \|$.

By the standard additional subdivision we get now

$$\| d^s(g \circ \psi) \| \leqq 1, \qquad s = 1, \ldots, k.$$

The images of ψ cover $A \supseteq S' \supseteq S$. Thus the reparametrizations ψ satisfy the conditions of Theorem 2.1, and their number does not exceed $\tilde{c}(k, l, m) \cdot M^{l/k} \cdot \mu(k, l, m)(\log M)^{\nu(k,l,m)}$. Theorem 2.1 is proved.

Added in proof. Recently M. Gromov and the author improved the result of Theorems 1, 2 and 3, obtaining the bound for $\kappa(\Psi)$, depending only on $d(u)$ and k (see [2]). The detailed proof will appear separately.

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